HOMEWORK 9 - ANSWERS TO (MOST) PROBLEMS

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Section 4.2: The Mean Value Theorem

4.2.6. $f(0) = f(\pi) = 0$, but $f'(x) = \sec^2(x) > 0$. This does not contradict Rolle's Theorem because f is not continuous on $[0, \pi]$ (it is discontinuous at $\frac{\pi}{2}$.

4.2.13. f is differentiable on $[0, 3]$; $c = -\frac{1}{2}\ln\left(\frac{1-e^{-6}}{6}\right)$ (you get this by solving $-2e^{-2c} = \frac{e^{-6}-1}{3}$.

4.2.18. Let $f(x) = 2x - 1 - \sin(x)$

At least one root: $f(0) = -1 < 0$ and $f(\pi) = 2\pi - 1 > 0$ and f is continuous, so by the **Intermediate Value Theorem (IVT)** the equation has at least one root.

At most one root: Suppose there are two roots a and b. Then $f(a) = f(b) = 0$, so by **Rolle's Theorem** there is at least one $c \in (a, b)$ such that $f'(c) = 0$. But $f'(c) = 2 - \cos(c) \neq 0$, which is a contradiction, and hence the equation has at most one root.

4.2.23. By the MVT, $\frac{f(4)-f(1)}{4-1} = f'(c)$ for some c in $(1, 4)$. Solving for $f(4)$ and using $f(1) = 10$, we get $f(4) = 3f'(c) + 10 \geq 6 + 10 = 16$.

4.2.29. This is equivalent to showing:

$$\left|\frac{\sin(a) - \sin(b)}{a - b}\right| \leq 1$$

Which is the same as:

$$\left|\frac{\sin(b) - \sin(a)}{b - a}\right| \leq 1$$

Which is the same as:

$$-1 \leq \frac{\sin(b) - \sin(a)}{b - a} \leq 1$$

But by the MVT applied to $f(x) = \sin(x)$, we get:

$$\frac{\sin(b) - \sin(a)}{b - a} = \cos(c)$$

for some c in (a, b). However, $-1 \leq \cos(c) \leq 1$, and so we're done!

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4.2.32. Let $f(x) = 2\sin^{-1}(x)$, $g(x) = \cos^{-1}(1-2x^2)$.

Then $f'(x) = \frac{2}{\sqrt{1-x^2}}$ and:

$$g'(x) = -\frac{-4x}{\sqrt{1-(1-2x^2)^2}} = \frac{4x}{\sqrt{1-1+4x^2-4x^4}} = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}} = f'(x)$$

(you get this by factoring out $4x^2$ out of the square root. This works because $x \geq 0$)

Hence $f'(x) = g'(x)$, so $f(x) = g(x) + C$

To get C, plug in $x = 0$, so $f(0) = g(0) + C$. But $f(0) = g(0) = 0$, so $C = 0$, whence $\boxed{f(x) = g(x)}$

4.2.34. Let $f(t)$ be the speed at time t. By the MVT with $a = 2:00$ and $b = 2:10$, we get:

$$\frac{f(2:10) - f(2:00)}{2:10 - 2:00} = f'(c)$$

But $2:10 - 2:00 = 10$ minutes $= \frac{1}{6}$ h, so:

$$\frac{50-30}{\frac{1}{6}} = f'(c)$$

Whence: $\boxed{f'(c) = 120}$ for some c between $2:00$ pm and $2:10$ pm. But $f'(c)$ is the acceleration at time c, and so we're done!

Section 4.3: How derivatives affect the shape of a graph

4.3.9.

(a) $f'(x) = 6x^2 + 6x - 36 = 6(x-2)(x+3)$; $\nearrow$ on $(-\infty, -3) \cup (2, \infty)$, $\searrow$ on $(-3, 2)$

(b) Local max: $f(-3) = 81$; Local min: $f(2) = -44$

(c) $f''(x) = 12x + 6$; CU on $(-\frac{1}{2}, \infty)$, CD on $(-\infty, \frac{-1}{2})$, IP $(-\frac{1}{2}, f(-0.5) = \frac{37}{2})$

4.3.13.

(a) $f'(x) = \cos(x) - \sin(x)$; $\nearrow$ on $(0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, \infty)$, $\searrow$ on $(\frac{\pi}{4}, \frac{5\pi}{4})$

(b) Local max: $f(\frac{\pi}{4}) = \sqrt{2}$; Local min: $f(\frac{5\pi}{4}) = -\sqrt{2}$

(c) $f''(x) = -\sin(x) + \cos(x)$; CU on $(\frac{3\pi}{4}, \frac{7\pi}{4})$, CD on $(0, \frac{3\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi)$, IP $(\frac{3\pi}{4}, 0), (\frac{7\pi}{4}, 0)$

4.3.43.

(a) $f'(\theta) = -2\sin(\theta) - 2\cos(\theta)\sin(\theta) = -2\sin(\theta)(1 + \cos(\theta))$; $\nearrow$ on $(\pi, 2\pi)$, $\searrow$ on $(0, \pi)$

(b) Local min: $f(\pi) = -1$, no local max.

(c) $f''(x) = -2\cos(\theta) + 2\sin^2(\theta) - 2\cos^2(\theta) = -2\cos(\theta) + 2 - 4\cos^2(\theta) = -4(\cos^2(\theta) - \frac{\cos(\theta)}{2} - \frac{1}{2}) = -4(\cos(\theta) + 1)(\cos(\theta) - \frac{1}{2})$; CU on $(\frac{\pi}{3}, \frac{5\pi}{3})$, CD on $(0, \frac{\pi}{3}) \cup (\frac{5\pi}{3}, 2\pi)$, IP $(\frac{\pi}{3}, \frac{5}{4}), (\frac{5\pi}{3}, \frac{5}{4})$

4.3.45.

(a) Vertical Asymptotes: $x = -1$, $x = 1$; Horizontal Asymptote: $y = 1$ (at $\pm\infty$)
(b) $f'(x) = \frac{-2x}{(x^2-1)^2}$; $\nearrow$ on $(-\infty, -1) \cup (-1, 0)$, $\searrow$ on $(0, 1) \cup (1, \infty)$ (remember that f is not defined at ± 1)
(c) Local max: $f(0) = 0$; No local min
(d) $f''(x) = \frac{-2(x^2-1)^2+(2x)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{6(x^2-1)(x^2+\frac{1}{3})}{(x^2-1)^4}$; CU on $(-\infty, -1) \cup (1, \infty)$, CD on $(-1, 1)$, No inflection points.

4.3.51.

(a) Vertical Asymptote: $x = -1$; Horizontal Asymptote: $y = 1$ (at $\pm\infty$)
(b) $f'(x) = \frac{1}{(x+1)^2} e^{-\frac{1}{x+1}}$; $\nearrow$ on $(-\infty, -1) \cup (-1, \infty)$ (remember f is not defined at -1)
(c) No local max/min
(d) $f''(x) = \frac{-2}{(x+1)^3} e^{-\frac{1}{x+1}} + \frac{1}{(x+1)^4} e^{-\frac{1}{x+1}} = \frac{-2x-1}{(x+1)^4} e^{-\frac{1}{x+1}}$; CU on $(-\infty, -1) \cup (-1, -\frac{1}{2})$, CD on $(-\frac{1}{2}, \infty)$, IP $= (-\frac{1}{2}, \frac{1}{e^2}$.

Section 4.4: L'Hopital's Rule

4.4.3.

(a) No, $-\infty$
(b) Yes, $\infty - \infty$
(c) No, ∞

4.4.4.

(a) Yes, 0^0
(b) No, 0
(c) Yes, 1^∞
(d) Yes, ∞^0
(e) No, ∞
(f) Yes, ∞^0

4.4.11.

$$\lim_{t \to 0} \frac{e^t - 1}{t^3} = \lim_{t \to 0} \frac{e^t}{3t^2} = \lim_{t \to 0} \frac{e^t}{6t} = \lim_{t \to 0} \frac{e^t}{6} = \frac{1}{6}$$

4.4.15.

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0$$

4.4.23.

$$\lim_{x \to 0} \frac{\tanh(x)}{\tan(x)} = \lim_{x \to 0} \frac{sech^2(x)}{\sec^2(x)} = 1$$

4.4.27.

$$\lim_{x \to 0} \frac{\sin^{-1}(x)}{x} = \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$$

4.4.29.

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}$$

4.4.40.

$$\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$$

4.4.47.

$$\lim_{x\to1} \frac{x}{x-1} - \frac{1}{\ln(x)} = \lim_{x\to1} \frac{x\ln(x)-(x-1)}{(x-1)\ln(x)} = \lim_{x\to1} \frac{\ln(x)+1-1}{\ln(x)+1-\frac{1}{x}} = \lim_{x\to1} \frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{2}$$

4.4.51.

$$\lim_{x\to\infty} x - \ln(x) = \lim_{x\to\infty} x(1-\frac{\ln(x)}{x}) = \infty \times (1-0) = \infty$$

4.4.56.

1) Let $y = \left(1+\frac{a}{x}\right)^{bx}$
2) $\ln(y) = bx\ln(1+\frac{a}{x})$
3)

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} bx\ln(1+\frac{a}{x}) = \lim_{x\to\infty} \frac{\ln(1+\frac{a}{x})}{\frac{1}{bx}} = \lim_{x\to\infty} \frac{(\frac{1}{1+\frac{a}{x}})(-\frac{a}{x^2})}{(-\frac{1}{x^2})(\frac{1}{b})} = \lim_{x\to\infty} \frac{ab}{1+\frac{a}{x}} = ab$$

4) So $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx} = e^{ab}$

4.4.59.

1) Let $y = x^{\frac{1}{x}}$
2) Then $\ln(y) = \frac{\ln(x)}{x}$
3) So $\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(x)}{x} = 0$
4) Hence $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = e^0 = 1$